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LETTER TO THE EDITOR

Comment on 'An interesting relation involving 3-j symbols'

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Abstract. It has been proved that the second relation involving 3-j symbols, denoted by $\bar{S}_l$ in Morgan's letter, vanishes for any natural number l.

In a recent letter, Morgan (1975) stated two relations concerning 3-j symbols:

$$S_l = \sum_{l'=0}^{l} \frac{1}{2l'-1} \begin{pmatrix} l & l' & l-l' \\ 0 & 0 & 0 \end{pmatrix}^2 = -\delta_{l0} \tag{1}$$

and

$$\bar{S}_l = \sum_{l'=0}^{l} \frac{1}{2l'+3} \begin{pmatrix} l & l' & l-l' \\ 0 & 0 & 0 \end{pmatrix}^2 - \sum_{l'=0}^{l} \frac{1}{2l'+1} \begin{pmatrix} l & l'+1 & l-l'+1 \\ 0 & 0 & 0 \end{pmatrix}^2. \tag{2}$$

Only the 'pseudo-orthogonality' relation S_l has been proved by Morgan (1975). However, due to the fact that the $\bar{S}_l$ vanish for $l=0, 1, 2, 3, 4$, one is tempted to conjecture that this second relation vanishes for any natural number l. This statement can be simply proved.

Let us start with the first sum in (2) and separate the $l'=l$ term from the rest:

$$A_l \equiv \sum_{l'=0}^{l} \frac{1}{2l'+3} \begin{pmatrix} l & l' & l-l' \\ 0 & 0 & 0 \end{pmatrix}^2 = \frac{1}{(2l+3)(2l+1)} + \sum_{l'=0}^{l-1} \frac{1}{2l'+3} \begin{pmatrix} l & l' & l-l' \\ 0 & 0 & 0 \end{pmatrix}^2,$$

which can be written for $l \neq 0$ as:

$$A_l = \frac{1}{(2l+3)(2l+1)} + \sum_{l'=1}^{l} \frac{1}{2l'+1} \begin{pmatrix} l & l'-1 & l-l'+1 \\ 0 & 0 & 0 \end{pmatrix}^2. \tag{3}$$

It is readily verified (de-Shalit and Talmi 1963) that for $1 \leqslant l' \leqslant l$,

$$\begin{pmatrix} l & l'-1 & l-l'+1 \\ 0 & 0 & 0 \end{pmatrix}^2 = \frac{l'(2l-2l'+1)}{(2l'-1)(l-l'+1)} \begin{pmatrix} l & l' & l-l' \\ 0 & 0 & 0 \end{pmatrix}^2. \tag{4}$$

In the case $l'=0$ the left-hand side of (4) seems meaningless, while the right-hand side is zero. However, by applying one of the symmetries discovered by Regge (1958), the 3-j symbol at the left can be formally written as (for $l'=0$):

$$\begin{pmatrix} l & \tfrac{1}{2}l & \tfrac{1}{2}l \\ -l-2 & \tfrac{1}{2}l+1 & \tfrac{1}{2}l+1 \end{pmatrix}.$$

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This symbol clearly vanishes, since the quantities in the lower row, representing the projections on the z axis of the corresponding quantities in the upper row, are for every value of l larger in absolute value than the above ones.

Inserting (4) into (3) and adding the vanishing $l'=0$ term to the sum, one gets

$$A_l=\frac{1}{(2l+3)(2l+1)}+\sum_{l'=0}^{l}\frac{1}{2l'+1}\frac{l'(2l-2l'+1)}{(2l'-1)(l-l'+1)}\begin{pmatrix} l & l' & l-l' \\ 0 & 0 & 0 \end{pmatrix}^2. \tag{5}$$

We now add $0=\beta_l S_l$ (β_l being an arbitrary quantity, eventually depending on l) to the right-hand side of (5) to obtain, still considering the case $l\neq 0$

$$A_l=\frac{1}{(2l+3)(2l+1)}+\sum_{l'=0}^{l}\frac{(1+\beta_l)l'(2l-2l'+1)+\beta_l(1+l)}{(2l'+1)(2l'-1)(l-l'+1)}\begin{pmatrix} l & l' & l-l' \\ 0 & 0 & 0 \end{pmatrix}^2. \tag{6}$$

Since β_l is completely arbitrary, we can choose it equal to -1, and (6) can be written as:

$$\begin{aligned}\frac{(2l+3)A_l}{l+1}&=\frac{1}{(2l+1)(l+1)}-\sum_{l'=0}^{l}\frac{2l+3}{(2l'+1)(2l'-1)(l-l'+1)}\begin{pmatrix} l & l' & l-l' \\ 0 & 0 & 0 \end{pmatrix}^2\\ &=\frac{1}{(2l+1)(l+1)}-\sum_{l'=0}^{l}\left(\frac{1}{2l'-1}-\frac{1}{2l'+1}+\frac{1}{(2l'-1)(l-l'+1)}\right)\\ &\quad\times\begin{pmatrix} l & l' & l-l' \\ 0 & 0 & 0 \end{pmatrix}^2. \end{aligned}\tag{7}$$

The first term in the sum vanishes due to (1), and (7) can be written, taking into account the original expression of A_l, as:

$$\begin{aligned}\frac{2l+3}{l+1}\sum_{l'=0}^{l}\frac{1}{2l'+3}\begin{pmatrix} l & l' & l-l' \\ 0 & 0 & 0 \end{pmatrix}^2-\sum_{l'=0}^{l}\frac{1}{2l'+1}\begin{pmatrix} l & l' & l-l' \\ 0 & 0 & 0 \end{pmatrix}^2\\ =\frac{1}{(2l+1)(l+1)}-\sum_{l'=0}^{l}\frac{1}{(2l'-1)(l-l'+1)}\begin{pmatrix} l & l' & l-l' \\ 0 & 0 & 0 \end{pmatrix}^2. \end{aligned}\tag{8}$$

Using expression (10) of Morgan's letter,

$$\begin{aligned}\frac{1}{(2l+1)(2l+3)}&=\sum_{l'=0}^{l}\frac{1}{(l'+1)(2l-2l'-1)}\begin{pmatrix} l & l'+1 & l-l'+1 \\ 0 & 0 & 0 \end{pmatrix}^2\\ &=\sum_{l'=0}^{l}\frac{1}{(2l'-1)(l-l'+1)}\begin{pmatrix} l & l'+1 & l-l'+1 \\ 0 & 0 & 0 \end{pmatrix}^2,\end{aligned}$$

(which is only valid for $l\neq 0$), and knowing that (de-Shalit and Talmi 1963):

$$\begin{pmatrix} l & l'+1 & l-l'+1 \\ 0 & 0 & 0 \end{pmatrix}^2=\frac{l+1}{2l+3}\begin{pmatrix} l & l' & l-l' \\ 0 & 0 & 0 \end{pmatrix}^2, \tag{9}$$

one immediately observes that the right-hand side of (8) is zero. One can note that the left-hand side of (8) is nothing else than $\bar{S}_l$, if one applies (9) in reversed order to its second term. So we can conclude that the $\bar{S}_l$ vanish for every natural number l.

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